Proportional fairness – a query

In *Charging and rate control for elastic traffic*, proportional fairness is defined: "A vector of rates $x = (x_s,\ s \in S)$ is *proportionally fair* if it is feasible (that is $x > 0$ and $Ax \le C$) and if for any other feasible vector x^*, the aggregate proportion of changes is negative: $\sum_{s \in S} \frac{x_s^* - x_s}{x_s} < 0$" [1]. ($S$ is defined as the set of source-sinks on the network, and the condition $Ax \le C$ stipulates that no resource's capacity should be exceeded.)

However, I have found that for certain networks there is no set of rates x that passes this test. On the other hand, I have found that, for those same networks, it *is* possible to find (analytically) a set of rates, x, that satisfy a dual equation to the one above: $\sum_{s \in S} \frac{x_s - x_s^*}{x_s^*} > 0$.

This implies that this set of rates, x, is *better* (in terms of proportional fairness) than *any* other feasible set of rates, x^*, that could be considered. And yet, this set of rates *does not meet* the criterion for proportional fairness – there are a whole class of alternative sets of rates, x^*, for which $\sum_{s \in S} \frac{x_s^* - x_s}{x_s} = 0$.

My main question, therefore, is: is the definition of proportional fairness correct or complete? In the cases I have considered, the set of rates which I believe to be the proportionally fair set always satisfies a modified version of the original equation: $\sum_{s \in S} \frac{x_s^* - x_s}{x_s} \le 0$, as well as satisfying the dual equation: $\sum_{s \in S} \frac{x_s - x_s^*}{x_s^*} > 0$. Should the definition of proportional fairness reflect either of these facts in some way?

My second question is more philosophical. In the networks I've considered a set of rates, x, can be found that satisfies $\sum_{s \in S} \frac{x_s - x_s^*}{x_s^*} > 0$ for any feasible alternative x^*, and yet there is a whole class of these alternatives for which $\sum_{s \in S} \frac{x_s^* - x_s}{x_s} = 0$ – which implies that these sets of rates, x^*, are *no worse* (in terms of proportional fairness) than the set x. This seems odd to me – x is **better** than x^*, but x^* is **no worse** than x. Why should this be?

Included below are the calculations that I have been through as I have examined these issues. If a network topology was such that Lemmas 1 & 2 both held (it may only be necessary for Lemma 1 to hold, I'm not sure), then it is not possible to find a set of rates that meets the defined criterion for proportional fairness, and yet it is possible to find one that is *better* than any feasible alternative. In order to keep the calculations simple, I have considered what is perhaps the simplest network for which these Lemmas hold. However, it is not difficult to see how the calculations would be extended if the network topology was changed (to another for which both Lemmas also held).

Proportional fairness – a query

Consider the simple network below, with three users 1, 2 and 3 and two resources, *a* and *b*. Let the throughput capacities of *a* and *b* be 1 unit, and let the flows generated by 1,2 and 3 be x_1, x_2 and x_3 respectively.

Then, assuming that the users all express the same willingness to pay (m_1=m_2=$m3$) then a proportionally fair set of rates $x = \{x_1, x_2, x_3\}$ must satisfy[1]

$$\sum_{s\in S} \frac{x_s^* - x_s}{x_s} < 0 \qquad (1)$$

for any other feasible set of rates x^*.

<u>Lemma 1</u>
In this case, if $x = \{x_1, x_2, x_3\}$ is proportionally fair, then both resources *a* and *b* will be fully utilised, and $x_2 = x_3$.

<u>Proof</u>
Assume $x = \{x_1, x_2, x_3\}$ is proportionally fair and one of the resources is not fully utilised. Assume that resource *a* is not fully utilised. In that case, flow x_2 can be increased by an amount $\delta = 1 - (x_1 + x_2) > 0$ giving $x^* = \{x_1, x_2 + \delta, x_3\}$, and:

$$\begin{aligned}\sum_{s\in S} \frac{x_s^* - x_s}{x_s} &= \frac{x_1 - x_1}{x_1} + \frac{x_2 + \delta - x_2}{x_2} + \frac{x_3 - x_3}{x_3} \\ &= \frac{\delta}{x_2} \\ &> 0\end{aligned}$$

which contradicts the assumption that $\{x_1, x_2, x_3\}$ satisfies equation (1).

Similarly, if resource *b* were not fully utilised, flow x_3 could be increased by an amount $\delta = 1 - (x_1 + x_3) > 0$, and equation (1) would again not be satisfied.

Hence both resources must be fully utilised if $x = \{x_1, x_2, x_3\}$ is proportionally fair, and $x_2 = x_3 = 1 - x_1$.

<u>Lemma 2</u>
If $x = \{x_1, x_2, x_3\}$ is such that both resources are fully utilised and equation (1) holds for all $x^* = \{x_1 + \delta, x_2 - \delta, x_3 - \delta\}$ for all $\delta \in [-x_1, 1 - x_1]$ then x is proportionally fair.

[1] Kelly, F.P. (1997). Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, **8**, 33-37.

Proof

By lemma 1, if $x = \{x_1, x_2, x_3\}$ is proportionally fair, then resources a and b will be fully utilised. Therefore the set of sets of feasible rates against which a set of feasible rates has to be tested to see if it is proportionally fair is just those which fully utilise both resources.

Assume that $y = \{y_1, y_2, y_3\}$ utilises both a and b fully. Let $\delta = y_1 - x_1$, then since both x and y are feasible, $\delta \quad [-x_1, 1 - x_1]$. Also, since $x_2 = x_3 = 1 - x_1$ and $y_2 = y_3 = 1 - y_1$,
$y_2 = 1 - (\delta + x_1) = (1 - x_1) - d = x_2 - \delta$ and, similarly, $y_3 = x_3 - \delta$. So, all sets of feasible rates which fully utilise both a and b can be written as $x^* = \{x_1 + \delta, x_2 - \delta, x_3 - \delta\}$ for some
$\delta \quad [-x_1, 1 - x_1]$, as required.

Now, assume that a proportionally fair set of rates exists, and denote that set by $x = \{x_1, x_2, x_3\}$. So, we have:

$$
\begin{aligned}
&\sum_{s\in S} \frac{x_s^* - x_s}{x_s} < 0 \\
\Leftrightarrow\ & \frac{x_1 + \delta - x_1}{x_1} + \frac{x_2 - \delta - x_2}{x_2} + \frac{x_3 - \delta - x_3}{x_3} < 0; \forall \delta \in \left[-x_1, 1 - x_1\right] \\
\Leftrightarrow\ & \frac{\delta}{x_1} - \frac{2\delta}{x_2} < 0 \\
\Leftrightarrow\ & \frac{x_1}{\delta} > \frac{x_2}{2\delta}
\end{aligned}
$$

Now, if $\delta \in [-x_1, 0)$: $$\frac{x_1}{\delta} > \frac{x_2}{2\delta} \Leftrightarrow x_1 < \frac{x_2}{2} \tag{2}$$

and if $\delta \in (0, 1 - x_1]$ $$\frac{x_1}{\delta} > \frac{x_2}{2\delta} \Leftrightarrow x_1 > \frac{x_2}{2} \tag{3}$$

Equations (2) and (3) cannot both be true simultaneously, and yet they both follow from equation (1). This seems to imply that there is no proportionally fair set of rates for this simple network.

However, if equation (1) is modified:

$$\sum_{s\in S} \frac{x_s^* - x_s}{x_s} \le 0 \tag{1a}$$

then the chain of inequalities also become $\le$ or $\ge$ and equations (2) and (3) become

$$\frac{x_1}{\delta} \ge \frac{x_2}{2\delta} \Leftrightarrow x_1 \le \frac{x_2}{2} \tag{2a}$$

and $$\frac{x_1}{\delta} \ge \frac{x_2}{2\delta} \Leftrightarrow x_1 \ge \frac{x_2}{2} \tag{3a}$$

respectively, indicating that the proportionally fair solution is given by:

$$x_1 = \frac{x_2}{2} \tag{4}$$

If we take $x = \left\{\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\}$ to satisfy this relationship, then for all sets of rates that fully utilise both resources, $\sum_{s\in S} \frac{x_s^* - x_s}{x_s} = 0$ – which seems to indicate that x is no better than any x^*.

On the other hand, consider the inequality

$$\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} > 0 \qquad (5)$$

as a possible test for proportional fairness.

In the case of the simple network under consideration, we have:

$$\begin{aligned}
\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} &= \frac{x_1-(x_1+\delta)}{x_1+\delta} + 2\frac{x_2-(x_2-\delta)}{x_2-\delta}\\
&= \frac{-\delta}{x_1+\delta} + \frac{2\delta}{x_2-\delta}\\
&= \frac{-\delta(x_2-\delta)+2\delta(x_1+\delta)}{(x_1+\delta)(x_2-\delta)}\\
&= \frac{-\delta x_2+\delta^2+2\delta x_1+2\delta^2}{(x_1+\delta)(x_2-\delta)}\\
&= \frac{3\delta^2+\delta(2x_1-x_2)}{(x_1+\delta)(x_2-\delta)}
\end{aligned}$$

Now, $\delta \in [-x_1, 1-x_1]$, i.e. $\delta \in [-x_1, x_2]$. So $(x_1+\delta)(x_2-\delta) \geq 0$ for any valid δ. So, therefore,

$$\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} > 0 \Leftrightarrow 3\delta^2 + \delta(2x_1 - x_2) > 0 \ \forall \delta \in [-x_1, x_2]$$

Clearly, if $2x_1 = x_2$, then this will be satisfied for all $\delta \neq 0$ ($\delta = 0 \in x = x^*$). If however, $2x_1 \neq x_2$, then consider the two cases $2x_1 > x_2$ and $2x_1 < x_2$. Recall that $x_2 = 1 - x_1$.

Case 1: $2x_1 > x_2$

$$\begin{aligned}
\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} > 0 &\Leftrightarrow 3\delta^2 + \delta(2x_1 - x_2) > 0 \ \forall \delta \in [-x_1, x_2]\\
&\Leftrightarrow 3\delta^2 + \delta(3x_1 - 1) > 0 \ \forall \delta \in [-x_1, 1-x_1]\\
&\Leftrightarrow \delta^2 + \delta(x_1 - \tfrac{1}{3}) > 0 \ \forall \delta \in [-x_1, 1-x_1]
\end{aligned}$$

let $x_1 - \frac{1}{3} = \alpha$ (> 0) then

$$\begin{aligned}
\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} > 0 &\Leftrightarrow \delta^2 + \alpha\delta > 0 \ \forall \delta \in [-x_1, x_2]\\
&\Leftrightarrow \delta(\delta + \alpha) > 0 \ \forall \delta \in [-x_1, 1-x_1]\\
&\Leftrightarrow \begin{cases} \delta > 0 \text{ and } \delta > -\alpha \\ \text{or} \\ \delta < 0 \text{ and } \delta < -\alpha \end{cases}
\end{aligned}$$

But $-\alpha = \frac{1}{3} - x_1 > -x_1$ and $-\alpha < 1 - x_1$ so

$(-\alpha, 0) \cap [-x_1, 1-x_1] = (-\alpha, 0) \cap [-x_1, x_2] \neq \varnothing$

Hence $\forall x_1 > \frac{1}{3}\ \exists \delta \in [-x_1, x_2]$ s.t. $3\delta^2 + \delta(2x_1 - x_2) < 0$

Case 2: $2x_1 < x_2$

As before,

$$\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} > 0 \Leftrightarrow \delta^2 + \delta\left(x_1 - \tfrac{1}{3}\right) > 0 \;\forall \delta \in \left[-x_1, 1 - x_1\right]$$

let $x_1 - \frac{1}{3} = \beta \; (< 0)$ then

$$\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} > 0 \Leftrightarrow \delta^2 + \beta\delta > 0 \;\forall \delta \in \left[-x_1, x_2\right]$$

$$\Leftrightarrow \delta\left(\delta + \beta\right) > 0 \;\forall \delta \in \left[-x_1, 1 - x_1\right]$$

$$\Leftrightarrow \begin{cases} \delta > 0 \text{ and } \delta > -\beta \\ \text{or} \\ \delta < 0 \text{ and } \delta < -\beta \end{cases}$$

But $-\beta = \frac{1}{3} - x_1 > -x_1$ and $-\beta < 1 - x_1$ so

$$\left(0, -\beta\right) \cap \left[-x_1, 1 - x_1\right] = \left(0, -\beta\right) \cap \left[-x_1, x_2\right] \neq \varnothing$$

Hence $\forall x_1 < \frac{1}{3} \; \exists \delta \in \left[-x_1, x_2\right]$ s.t. $3\delta^2 + \delta\left(2x_1 - x_2\right) < 0$

Therefore, for all pairs x_1, x_2 with $2x_1 \neq x_2$, $\exists \delta \in \left[-x_1, x_2\right]$ s.t. $3\delta^2 + \delta\left(2x_1 - x_2\right) < 0$.

And so, equation (5) can be used to derive the proportionally fair solution analytically (for this simple network), yielding $x = \left\{\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\}$, and $\sum_{s\in S}\frac{x_s - x_s^*}{x_s^*} > 0$ for all feasible x^* which fully utilise both resources – indicating that x really is better than all alternatives, x^*.